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On cone perturbing Liapunov function for impulsive differential systems

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Abstract

The concept of ϕ_0-stability and ϕ_0-boundedness recently were introduced for systems of ordinary differential equations (ODEs). Perturbing Liapunov function method was discussed for systems of ODEs and extend to systems of functional differential equations (FDEs). In this paper, we extend these notions to impulsive systems of differential equations via cone perturbing Liapunov function method.

1. Introduction

The problem of qualitative mathematical theory of impulsive systems of differential equations has been interest by a great numbers of mathematicians Bainov and Simeonov [2,3], Kulev and Bainov [4], Lakshmikantham et al. [7], Somoilenko and Perslyuk [8]. Furthermore these systems are adequate mathematical models for numerous processes and phenomena studied in biology, physics technology, etc.

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The main purpose of this paper is to extend the notion of ϕ_0-stability and ϕ_0-boundedness to impulsive systems of differential equations via cone perturbing Liapunov function. This method is also extend of both perturbing Liapunov function of [5], and cone-valued Liapunov function method of [6], for a cone in cone-valued Liapunov function method is unit element or Liapunov function is unperturbed function, this method becomes Liapunov direct method, and for the motivation of this work is the recent work of [1,4,5].

Let $\Re^s$ be an s-dimensional Euclidean space with a suitable norm $\|\cdot\|$. Let $\Re^+ = [0,\infty)$. Define

$$S^i(\rho) = \{x \in \Re^j : \|x\| < \rho,\ \rho > 0\}.$$

Consider the system of differential equations with impulses

$$\left.\begin{aligned} x' &= f(t,x) + h(t,y), \quad t \neq \tau_i(x,y), \quad \Delta x|_{t=\tau_i(x,y)} = A_t(x) + B_t(y), \\ y' &= F(t,x,y), \qquad\qquad t \neq \tau_i(x,y), \quad \Delta y|_{t=\tau_i(x,y)} = C_t(x,y), \end{aligned}\right\} \tag{1.1}$$

where $x \in \Re^n$, $y \in \Re^m$, $f : \Re^+ \times S^n(\rho) \to \Re^n$, $h : \Re^+ \times S^m(\rho) \to \Re^m$, $F : \Re^+ \times S^n(\rho) \times S^m(\rho) \to \Re^m$, $A_t : S^n(\rho) \to \Re^n$, $B_t : S^m(\rho) \to \Re^m$, $C_t : S^n(\rho) \times S^m(\rho) \to \Re^m$, $\tau_i : S^n(\rho) \times S^m(\rho) \to \Re^1$.

$$\Delta x|_{t=\tau(x,y)} = x(t+0) - x(t-0), \quad \Delta y|_{t=\tau(x,y)} = y(t+0) - y(t-0).$$

Let $x(t, t_0, x_0, y_0)$, $y(t, t_0, x_0, y_0)$ be solution of the system (1.1), satisfying the same initial values $x(t_0 + 0, t_0, x_0, y_0) = x_0$, $y(t_0 + 0, t_0, x_0, y_0) = y_0$ for $t_0 \in \Re^+$, $x_0 \in S^n(\rho)$, $y_0 \in S^m(\rho)$. The solution $(x(t), y(t))$ of the system (1.1) are piecewise continuous functions with points of discontinuity of the first type in which they are left continuous, i.e. at the moment t_i when the integral curve of the solution $(x(t), y(t))$ meets the hypersurface

$$\sigma_t = \{(t,x,y) \in \Re^+ \times S^n(\rho) \times S^m(\rho) : t = \tau_i(x,y)\}.$$

The following relations are satisfied:

$$x(t_i - 0) = x(t_i), \quad \Delta x|_{t=t_i} = A_t(x(t_i)) + B_t(y(t_i)),$$

$$y(t_i - 0) = y(t_i), \quad \Delta y|_{t=t_i} = C_t(x(t_i), y(t_i)),$$

together with system (2.1), we consider the following system with impulses:

$$x' = f(t,x), \quad t \neq \tau_i(x,0), \quad \Delta x|_{t=\tau_i(x,0)} = A_t(x). \tag{1.2}$$

Let

$$s_t = \{(t,x) \in \Re^+ \times S^n(\rho) : t = \tau_i(x,0)\}.$$

For any subset $E_1 \subset \Re^n$, we denote by $\overline{E}_1$, E_1^c, and ∂E_1 the closure, the complement, and the boundary of E_1 respectively. Furthermore for any subset

$E_2 \subset \Re^n$, we denote by $\overline{E}_2$, E_2^c, and ∂E_2 the closure, the complement, and the boundary of E_2 respectively, where $S = S_1 \cup S_2$ and $E = E_1 \cup E_2$.

The following definitions are depending on that given in [1,6].

Definition 1. A proper subset K_1 of $\Re^n$ or a proper subset K_2 of $\Re^m$ is called a cone if

(i) $\lambda K_i \subset K_i$, $\lambda \geqslant 0$,
(ii) $K_i + K_i \subset K_i$,
(iii) $\overline{K}_i = K_i$,
(iv) $K_i^\circ \neq \emptyset$,
(v) $K_i \cap (-K_i) = \{0\}$,

where $\overline{K}_i$ and K_i° denote the closure and interior of K_i respectively, and ∂K_i denotes the boundary of K_i, $i = 1, 2$, it follows that $K = K_1 \cup K_2 \subset \Re^n \cup \Re^m$ be a cone in $\Re^n \cup \Re^m$.

As in [1,5] we introduce the following definitions.

Definition 2. The set K^* is called the adjoint cone if

$$K^* = \{\phi \in \Re^n \cup \Re^m : (\phi, x + y) \geqslant 0 \quad \text{for } x \in K_1 \subset K, y \in K_2 \subset K\}$$

satisfies the properties (i)–(v) of Definition 1, where $(\phi, x + y) \leqslant \|\phi\|(\|x\| + \|y\|)$. For $m > n$, $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$. Thus

$$\begin{aligned} x + y &= (x_1, x_2, \ldots, x_n, , 0, 0, \ldots, 0) + (y_1, y_2, \ldots, y_m) \\ &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n, y_{n+1}, \ldots, y_m), \end{aligned}$$

$x \in \partial K_i$ iff $(\phi, x) = 0$, for some $\phi \in K_{i0}^*$, $K_{i0}^* = K_{i0} \setminus \{0\}$, $i = 1, 2$.

Definition 3. A function $g : D \to \Re^n$, $D \subset \Re^n$ is called quasimonotone relative to the cone K_i, $i = 1, 2$, if $x, y \in D$ and $y - x \in \partial K_i$, then there exists $\phi_0 \in K_{i0}^*$ such that $(\phi_0, y - x) = 0$ and $(\phi_0, g(y) - g(x)) \geqslant 0$.

Definition 4. A function $b(r)$ is said to be belong the class $\mathscr{K}$ if $a \in C[\Re^+, \Re^+]$, $b(0) = 0$, and $b(r)$ is strictly monotone increasing in r. Let $\tau_0(x, y) = 0$ for $(x, y) \in S^n(\rho) \times S^m(\rho)$.

Following [4] we define the sets

$$G_i = \{(t, x, y) \in \Re^+ \times S^n(\rho) \times S^m(\rho) : \tau_{i-1}(x, y) < t < \tau_i(x, y)\},$$

$$\Omega_i = \{(t, x) \in \Re^+ \times S^n(\rho) : \tau_{i-1}(x, 0) < t < \tau_i(x, 0)\}.$$

As in [4], we use the classes $\mathscr{V}_0$ and $\mathscr{W}_0$ of piecewise continuous functions which are analogue to Liapunov functions.

Definition 5 [4]. We say that the function $V : \Re^+ \times S^n(\rho) \times S^m(\rho) \to K$ belongs to the class $\mathscr{V}_0$ if the following conditions hold:

(1) The function V is continuous in $\bigcup_{i=1}^{\infty} G_i$ and is locally Lipschitzian with respect to x and y in each of the sets G_i, and $V(t, 0, 0) = 0$ for $t \in \Re^+$.

(2) $V(t_0 - 0, x_0, y_0) = V(t_0, x_0, y_0)$, for each $i = 1, 2, \ldots$ and for any point $(t_0, x_0, y_0) \in \sigma_t$ the following limits:

$$V(t_0 - 0, x_0, y_0) = \lim_{\substack{(t,x,y)\to(t_0,x_0,y_0)\\(t,x,y)\in G_i}} V(t,x,y),$$

$$V(t_0 + 0, x_0, y_0) = \lim_{\substack{(t,x,y)\to(t_0,x_0,y_0)\\(t,x,y)\in G_{i+1}}} V(t,x,y)$$

are exist.

(3) For any point $(t, x, y) \in \sigma_t$, the following inequality holds:

$$V(t+0, x + A_t(x) + B_t(y), y + C_t(x,y)) \leqslant V(t,x,y). \tag{1.3}$$

Definition 6 [4]. We say that the function $W : I \times S^n(\rho) \to K$ belongs to the class $\mathscr{W}_0$ if the following conditions hold:

(1) The function W is continuous in $\bigcup_1^{\infty} \Omega_i$ and is locally Lipschitz with respect to x in each of the sets Ω_i, $W(t_0 - 0, x_0) = W(t_0, x_0)$, $W(t, 0) = 0$, for $t \in \Re^+$, and the following limits:

$$W(t_0 - 0, x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in\Omega_i}} W(t,x),$$

$$W(t_0 + 0, x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in\Omega_i}} W(t,x)$$

are exist.

(2) For any point $(t, x) \in s_t$, the following inequality holds:

$$W(t+0, x + A_t(x)) \leqslant W(t,x). \tag{1.4}$$

Let $V \in \mathscr{V}_0$, and $x(t)$, $y(t)$ be a solution of (1.1), for $(t,x,y) \in \bigcup_1^{\infty} G_i$ following [4] we define

$$D^+V_1(t,x,y) = \lim_{s\to 0}[V(t+s, x + s(f(t,x) + h(t,y)), y + sF(t,x,y)) - V(t,x,y)],$$

and $V_1'(t,x,y) = D^+V(t,x,y)$, $t \neq \tau_i(x,y)$, where $D^+V(t,x,y)$ is the upper right Dini derivative of the function $V(t,x,y)$.

Analogously one can define the function $W'_{(1.2)}(t,x)$ for an arbitrary function $W \in \mathscr{W}_0$ for $(t,x) \in \bigcup_1^\infty \Omega_i$. The following definition is new and related with that of [1,4].

Definition 7. The zero solution of system (2.1) is said to be ϕ_0-equistable if for all $\epsilon > 0$, for all $t_0 \in \Re^+$, there exist $\delta = \delta(t_0, \epsilon) > 0$, for all $(x_0, y_0) \in (S^n(\rho) \times S^m(\rho))$ such that for $\phi_0 \in K_0^*$,

$$(\phi_0, x_0 + y_0) < \delta \text{ implies } (\phi_0, x^*(t,t_0,x_0,y_0) + y^*(t,t_0,x_0,y_0)) < \epsilon, \; t \geqslant t_0,$$

where x^* and y^* is the maximal solution of (1.1).

Definition 8. The zero solution of (1.1) is ϕ_0-bounded if for $\alpha > 0$, $t_0 \in \Re^+$, there exists $\beta(t,\alpha) > 0$ such that for $\phi_0 \in K_0^*$,

$$(\phi_0, x^* + y^*) < \alpha,$$

provided that $(\phi_0, x_0 + y_0) < \beta(t,\alpha)$.

Definition 9. We say conditions (A) hold if the following conditions are satisfied:

(A_1) The functions $f(t,x)$, $h(t,y)$, and $F(t,x,y)$ are continuous in their definitions domains and $f(t,x)$ is quasimonotone in x relative to the cones K_1, and $F(t,x,y)$ is quasimonotone in x and y relative to the cones K_1 and K_2, $f(t,0) = h(t,0) = 0$, $F(t,0,0) = 0$ for $t \in \Re^+$, and there exists a constant $L > 0$ such that

$$F(t,x,y) \leqslant L, \quad (t,x,y) \in \Re^+ \times S^n(\rho) \times S^m(\rho).$$

(A_2) There exists a continuous function $P : I \to I$ such that

$$P(0) = 0 \text{ and } \|h(t,y)\| \leqslant P(\|y\|) \quad \text{for } (t,x) \in \Re^+ \times S^n(\rho).$$

(A_3) The functions A_t, B_t, C_t are continuous in their definitions domains and $A_t(0) = B_t(0) = C_t(0,0) = 0$, and for $x \in S^n(\rho)$ and $y \in S^m(\rho)$, then

$$\|x + A_t(x) + B_t(y)\| \leqslant \|x\| \quad \text{and} \quad \|y + c_t(x,y)\| \leqslant \|y\|, \quad i = 1,2.$$

(A_4) The functions $\tau_i(x,y)$ are continuous and for $(x,y) \in S^n(\rho) \times S^m(\rho)$ the following relations hold: $0 < \tau_1(x,y) < \tau_2(x,y) < \cdots < \lim_{t\to\infty} \tau_i(x,y) = \infty$ uniformly in $S^n(\rho) \times S^m(\rho)$, and

$$\inf_{S^n(\rho)\times S^m(\rho)} \tau_{i+1}(x,y) - \sup_{S^n(\rho)\times S^m(\rho)} \tau_i(x,y) \geqslant \theta > 0, \quad i = 1,2,\ldots$$

(A_5) For each point $(t_0, x_0, y_0) \in \Re^+ \times S^n(\rho) \times S^m(\rho)$, the solution $x(t, t_0, x_0)$, $y(t, t_0, x_0, y_0)$ of the system (2.1) is unique and defined in (t_0, ∞).

(A_6) For each point $(t_0, x_0) \in \Re^+ \times S^n(\rho)$ the solution $x(t, t_0, x_0)$ of system (2.2) satisfying $x(t_0 + 0, t_0, x_0) = x_0$ is unique and exists for all $t \in (t_0, \infty)$.

(A_7) The integral curve of each solution of system (1.1) meets each of the hypersurfaces $\{\sigma_t\}$ at most once.

2. ϕ_0-boundedness

The following result discussed concept of ϕ_0-boundedness of the system (1.1) via cone perturbing Liapunov function

Theorem 1. *Let the conditions* (A) *be satisfied, and* $E_1, E_2 \subset \Re^n$ *be compact subsets. Suppose that there exist two functions* $V_1(t,x,y) \in C[\Re^+ \times (\overline{E}_1^c \cap \overline{E}_2^c), K]$, $V_2(t,x,y) \in C[\Re^+ \times S^n(\rho) \times S^m(\rho), K]$, *and there exist two functions* $g_1 \in C[\Re^+ \times \Re^+, K]$ $g_2 \in C[\Re^+ \times \Re^+, K]$ *with* $V_1(t, 0, 0) = V_2(t, 0, 0) = g_1(t, 0) = g_2(t, 0) = 0$ *such that*

(H_1) *$V_1(t, x, y)$ is bounded and Lipschitzian in x and y relative to the cones K_1 and K_2 respectively, and*

$$D^+V_1(t,x,y) \leqslant g_1(t, V_1(t,x,y)), \quad (t,x,y) \in \Re^+ \times \overline{E}_1^c \cap \overline{E}_2^c. \tag{2.1}$$

(H_2) *$V_2(t, x, y)$ is Lipschitzian with respect to x and y relative to the cones K_1 and K_2 respectively, and*

$$b(\phi_0, x^* + y^*) \leqslant (\phi_0, V_2(t,x,y)) \leqslant a(\phi_0, x^* + y^*), \tag{2.2}$$

where $a, b \in \mathscr{K}$, $(t,x,y) \in \Re^+ \times S^n(\rho) \times S^m(\rho)$.

(H_3) *For each* $(t,x,y) \in \Re^+ \times S^n(\rho) \times S^m(\rho)$,

$$D^+(\phi_0, V_1(t,x,y)) + D^+(\phi_0, V_2(t,x,y)) \leqslant g_2(t, V_1(t,x,y) + V_2(t,x,y)). \tag{2.3}$$

(H_4) *If the zero solution of the system differential equation*

$$u' = g_1(t,u), \quad u(t,u) \geqslant u_0 \tag{2.4}$$

is ϕ_0-bounded, and if the zero solution of the system differential equation

$$w' = g_2(t,w), \quad w(t_0) \geqslant w_0 \tag{2.5}$$

is uniformly ϕ_0-bounded.

Then the zero solution of (1.1) *is ϕ_0-bounded.*

Proof. Since E_1, E_2 are compact subsets of $\Re^n$, there exists $\rho > 0$ such that $S_0^n(\rho)$, $S_0^m(\rho) \subset (E_1 \cap E_2, \rho_0)$ for some $\rho_0 > 0$, where

$$S(E_1 \cap E_2, \rho_0) = \{x \in \Re^n : d(x_t, E_1 \cap E_2) < \rho_0\}$$

and

$$d(x, E_1 \cap E_2) = \inf_{y \in E_1 \cap E_2} \|x - y\|.$$

Let $t \in \Re^+$, $\alpha \leqslant \rho$ be given, and $\alpha_1 = \alpha_1(t_0, \alpha) = \max(\alpha_0, \alpha^*)$, where

$$\alpha_0 = \max\left[V_1(t_0, x_0, y_0) : x_0, y_0 \in \overline{S^n(\alpha) \cap S^m(\alpha)} \cap E^c\right]$$

and $\alpha^* \geqslant V_1(t, x, y)$, for $(t, x, y) \in \Re^+ \times \partial(E_1 \cap E_2)$, where $E^c = E_1^c \cup E_2^c$, and $E = E_1 \cup E_2$. Since the zero solution of (2.4) is ϕ_0-bounded, given $\alpha_1 > 0$ and $t_0 \in \Re^+$, there exists $\beta_0 = \beta_0(t, \alpha_1) > 0$ such that for $\phi_0 \in K_0^*$

$$(\phi_0, r_1(t, t_0, u_0)) < \alpha_1, \tag{2.6}$$

whenever $(\phi_0, u_0) < \beta_0$, where $r_1(t, t_0, u_0)$ is the maximal solution of (2.4).

Also, since the zero solution of (2.5) is uniformly ϕ_0-bounded, given $\alpha_2 > 0$, $t_0 \in \Re^+$, there exists $\beta_1(\alpha_2) > 0$ such that for $\phi_0 \in K_0^*$,

$$(\phi_0, r_2(t, t_0, u_0)) < \alpha_2 \tag{2.7}$$

provided that $(\phi_0, w_0) < \beta_1(\alpha_2)$, where $r_2(t, t_0, w_0)$ is the maximal solution of (2.5).

Now, we choose $u_0 = V_1(t_0, x_0, y_0)$, and $\alpha_2 = a(\alpha) + \beta_0$. As $b(u) \to \infty$ with $u \to \infty$, we can choose $\beta = \beta(t_0, \alpha)$ such that

$$b(\beta) > \beta_1(\alpha_2). \tag{2.8}$$

Now, to prove that the zero solution of (1.1) is ϕ_0-bounded, it must be shown that $(\phi_0, x_0 + y_0) \in S^n(\alpha) \cap S^m(\alpha)$ implies for the maximal solution $x^*(t, t_0, x_0)$, $y^*(t, t_0, x_0)$ satisfies $(\phi_0, x^*(t, t_0, x_0) + y^*(t, t_0, x_0)) \leqslant \beta(t_0, \alpha)$.

Suppose that this is not true, then there exists $t^* > t_0, \alpha > 0$, with $(\phi_0, x_0 + y_0) \in S^n(\alpha) \cap S^m(\alpha)$ such that for $\phi_0 \in K_0^*$,

$$\lim_{t \to \infty} (\phi_0, x^*(t^*, t_0, x_0), y^*(t, t_0, x_0)) = \beta.$$

Since $S(E, \rho) \subset S(\alpha)$, there are two possibilities to consider:

(I) $x(t, t_0, x_0), y(t, t_0, x_0) \in E^c$ for $t \in [t_0, t^*]$.

(II) There exists $t_2 \geqslant t_0$ such that

$$x(t, t_0, x_0) \in \partial E, \;\; x(t, t_0, x_0) \in E^c \quad \text{for } t \in [t_0, t^*].$$

If case (I) is true, then we can find $t_1 > t_0$ such that

$$\left.\begin{aligned}&(\phi_0, x^*(t_1,t_0,x_0) + y^*(t_1,t_0,x_0)) \in \partial(S_1(\alpha)\cap S_2(\alpha)),\\&(\phi_0, x^*(t^*,t_0,x_0) + y^*(t^*,t_0,x_0)) \in \partial(S_1(\beta)\cap S_2(\beta)),\\&(\phi_0, x^*(t,t_0,x_0) + y^*(t,t_0,x_0)) \in \partial(S_1(\alpha)\cap S_2(\alpha)), t\in[t_0,t^*].\end{aligned}\right\} \tag{2.9}$$

Setting

$$m(t) = V_1(t,x(t,t_0,x_0),y(t,t_0,x_0)) + V_2(t,x(t,t_0,x_0),y(t,t_0,x_0)), \quad t\in[t_1,t^*].$$

It is easy to obtain, from (2.3), and thus from [6],

$$D^+m(t) \leqslant g_2(t,m(t)), \quad t\in[t_1,t^*].$$

Consequently, by comparison Theorem 1.4.1 of [6], we get

$$(\phi_0, m(t)) \leqslant (\phi_0, r_2(t,t_1m(t))), \quad t\in[t_1,t^*], \phi_0\in K_0^*,$$

where $r_2(t, t_1, w_0)$ is the maximal solution of (2.5) such that $r_2(t_1, t_1, w_0) = w_0$. Thus

$$\begin{aligned}&V_1(t^*,x(t^*,t_0,x_0),y(t^*,t_0,x_0)) + V_2(t^*,x(t^*,t_0,x_0),y(t^*,t_0,x_0))\\&\quad\leqslant r_2(t^*,t_1,V_1(t_1,x(t_1,t_0,x_0),y(t_1,t_0,x_0)) + V_2(t_1,x(t_1,t_0,x_0),y(t_1,t_0,x_0))).\end{aligned} \tag{2.10}$$

Similarly, from (2.1) we have

$$V_1(t_1,x(t_1,t_0,x_0),y(t_1,t_0,x_0)) \leqslant r_1(t_1,t_0,V_1(t_0,x_0,y_0)),$$

and thus for $\phi_0\in K_0^*$,

$$(\phi_0, V_1(t_1,x(t_1,t_0,x_0),y(t_1,t_0,x_0))) \leqslant (\phi_0, r_1(t_1,t_0,V_1(t_0,x_0,y_0))), \tag{2.11}$$

where $r_1(t_1, t_0, u_0)$ is the maximal solution of (2.4).

From the fact that $u_0 = V_1(t_0, x_0, y_0) < \alpha_1$, and (2.6) yield

$$(\phi_0, r_1(t_1,t_0,V_1(t_0,x_0,y_0))) \leqslant \alpha_1 \quad \text{for } (\phi_0,u_0) < \beta_0. \tag{2.12}$$

Furthermore for $\phi_0\in K_0^*$,

$$(\phi_0, V_2(t_1,x(t_1,t_0,x_0),y(t_1,t_0,x_0))) \leqslant a(\alpha), \tag{2.13}$$

where $a(\alpha) = \|\phi_0\|\|a_3(\alpha)\|$, $a_3(\alpha) > 0$, from (2.12) and (2.13), we have

$$\begin{aligned}(\phi_0,w_0) &= (\phi_0, V_1(t_1,x(t_1,t_0,x_0),y(t_1,t_0,x_0)))\\&\quad + (\phi_0, V_2(t_1,x(t_1,t_0,x_0),y(t_1,t_0,x_0))) < a(\alpha) < \alpha_2.\end{aligned} \tag{2.14}$$

Hence, from (2.2), (2.7)–(2.10), (2.14), and the fact that $V_1 \geqslant 0$,

$$b(\beta) \leqslant \beta_1(\alpha_2) \leqslant b(\beta). \tag{2.15}$$

If the case (II) holds, we again arrive at the inequality (2.10), where $t_1 > t$ satisfies (2.9). Now, we have in place of (2.11) the inequality

$$V_1(t_1, x(t_1, t_0, x_0), y(t_1, t_0, x_0)) \leqslant r_1(t_1, t_2, V_1(t_2, x(t_2, t_0, x_0), y(t_2, t_0, x_0))).$$

Since $x(t_2, t_0, x_0)$, $y(t_2, t_0, x_0) \in \partial E$ and $V_1(t_2, x(t_2, t_0, x_0), y(t_2, t_0, x_0)) \leqslant \alpha^* \leqslant \alpha_1$. Thus, we get the same contradiction in (2.15). This proves that

$$(\phi_0, x^*(t, t_0, x_0) + y^*(t_1, t_0, x_0))\alpha_2,$$

whenever $(\phi_0, x_0 + y_0) < \beta_1(t_0, \alpha_2)$, and the proof is completed. □

3. ϕ_0-Equistability

In this section, we discuss the notion of ϕ_0-equistability property of the system of (1.1) via cone perturbing Liapunov function method. Define the following set:

$$Z = \{(x, y) : (x, y) \in (S^n(\rho) \cap S^n_c(\eta)) \times (S^m(\rho) \cap S^m_c(\eta))\}.$$

Theorem 2. *Let the conditions* (A) *be satisfied and assume that there exist two functions* $V_1(t, x, y) \in C[\Re^+ \times S^n(\rho) \times S^m(\rho), K], V_2(t, x, y) \in C[\Re^+ \times Z, K]$, *for any* $\eta > 0, V_1(t, 0, 0) - V_2(t, 0, 0) = 0$, *where* $S^j_c(\eta)$ *is the complement of* $S^j(\eta)$, $j = n, m$. *Further assume that there exist two functions* $g_1 \in C[\Re^+ \times \Re^+, K]; g_2 \in C[\Re^+ \times \Re^+, K]$ *with* $g_1(t, 0) = g_2(t, 0) = 0$ *such that*

(H$_5$) $V_1(t, x, y)$ *is bounded and Lipschitzian in* x *and* y *relative to the cones* K_1 *and* K_2 *respectively*, *and*

$$D^+(\phi_0, V_1(t, x, y)) \leqslant g_1(t, V_1(t, x, y)), \quad (t, x, y) \in \Re^+ \times S^n(\rho) \times S^m(\rho).$$

(H$_6$) $V_{2\cdot\eta}(t, x, y)$ is locally Lipschitzian in x and y relative to the cones K_1, and K_2 respectively, such that

$$a(\phi_0, x + y) \leqslant (\phi_0, V_{2\cdot\eta}(t, x, y)) \leqslant b(\phi_0, x + y), \ (t, x, y) \in \Re^+ \times S^n(\rho) \times S^m(\rho), \ a, b \in \mathscr{K}.$$

(H$_7$) For $(t, x, y) \in \Re^+ \times S^n(\rho) \times S^m(\rho)$,

$$D^+V_1(t, x, y) + D^+V_2(t, x, y) \leqslant g_2(t, V_1(t, x, y) + V_2(t, x, y)).$$

If the zero solution of the differential equations

$$u' = g_1(t, u), \quad u(t, t_0, u_0) = u_0 \geqslant 0 \tag{3.1}$$

is ϕ_0-equistable, and if the zero solution of

$$v' = g_2(t, v), \quad v(t, t_0, v_0) = v_0 \geqslant 0 \tag{3.2}$$

is uniformly ϕ_0-stable, then the zero solution of the system (2.1) is ϕ_0-equistable.

Proof. From our assumption, the zero solution of (2.5) is uniformly ϕ_0-stable, let $0 < \epsilon < \rho$, given $b(\epsilon) > 0$ and $t_0 \in \Re^+$, there exists $\delta_0 = \delta_0(\epsilon) > 0$ such that for $\phi_0 \in K_0^*$

$$(\phi_0, r_2(t, t_0, w_0)) < b(\epsilon), \quad t \geqslant t_0, \tag{3.3}$$

provided that $(\phi_0, w_0) < \delta$, where $r_2(t, t_0, w_0)$ is the maximal solution of (2.5).

From the condition (H_6), there exists $\delta_2 = \delta_2(\epsilon) > 0$ such that

$$a(\delta_0) < \frac{\delta}{2}. \tag{3.4}$$

From our assumption, the zero solution of (2.4) is ϕ_0-equistable, given $\frac{\delta}{2}$, and $t_0 \in \Re^+$, there exists $\delta^* = \delta^*(t_0, \epsilon) > 0$ such that for $\phi_0 \in K_0^*$

$$(\phi_0, r_1(t, t_0, u_0)) < \frac{\delta}{2}, \quad t \geqslant t_0, \tag{3.5}$$

provided that $(\phi_0, u_0) < \delta^*$, $r_1(t, t_0, u_0)$ being the maximal solution of (2.4).

Following [7], choose $u_0 = V_1(t_0, x_0, y_0)$, since $V_1(t, x, y)$ is continuous and $V_1(t, 0, 0) = 0$, there exists $\delta_1 > 0$ such that for $\phi_0 \in K_0^*$

$$(\phi_0, x_0) < \delta \Rightarrow (\phi_0, V(t_0, x_0, y_0)) < \epsilon^*, \quad t \geqslant t_0, \tag{3.6}$$

holds simultaneously, we set $\delta = \min(\delta_1, \delta_2)$.

Now, to prove that the zero solution of (1.1) is ϕ_0-equistable, i.e.,

$$(\phi_0, x_0 + y_0) < \delta \quad \Rightarrow \quad (\phi_0, x^* + y^*) < \epsilon, \quad t \geqslant t_0.$$

Suppose that this is not true, there exist $t_1, t_2 > t_0$ such that for $(\phi_0, x_0 + y_0) < \delta$,

$$\begin{cases} (\phi_0, x^*(t_1, t_0, x_0) + y^*(t_1, t_0, x_0)) \in \partial(S_1(\delta_2) \cap S_2(\delta_2)), \\ (\phi_0, x^*(t_2, t_0, x_0) + y^*(t_2, t_0, x_0)) \in \partial(S_1(\epsilon) \cap S_2(\epsilon)), \\ (\phi_0, x^*(t, t_0, x_0) + y^*(t, t_0, x_0)) \in S(\epsilon) \cap S^c(\delta_0), \quad t \in [t_1, t_2]. \end{cases} \tag{3.7}$$

Let $\delta_2 = \eta$. So that the condition (H_6) is assured. Setting

$$m(t) = V_1(t, x(t, t_0, x_0), y(t, t_0, x_0)) + V_{2\cdot\eta}(t, x(t, t_0, x_0), y(t, t_0, x_0)), \quad t \in [t_1, t_0],$$

we get for $\phi_0 \in K_0^*$,

$$D^+(\phi_0, m(t)) \leqslant g_2(t, m(t)), \quad t \in [t_1, t_2],$$

which yields

$$(\phi_0, V_1(t_2, x(t_2, t_0, x_0), y(t_2, t_0, x_0)) + V_{2\cdot\eta}(t_2, x(t_2, t_0, x_0), y(t_2, t_0, x_0)))$$
$$\leqslant (\phi, r_2(t_1, t_0, V_1(t_1, x(t_1, t_0, x_0), y(t_1, t_0, x_0)) + V_{2\cdot\eta}(t_1, x(t_1, t_0, x_0), y(t_1, t_0, x_0)))),$$

where $r_2(t_1, t_0, w_0)$ is the maximal solution of (2.5), and $r_2(t_1, t_0, w_0) = w_0$. Also, we have for $\phi_0 \in K_0^*$,

$$(\phi_0, V_1(t_1, x(t_1, t_0, x_0), y(t_1, t_0, x_0))) \leqslant (\phi_0, r_1(t_1, t_0, V_1(t_0, x_0, y_0))),$$

where $r_1(t_1, t_0, u_0)$ is the maximal solutions of (2.4).

By (3.5) and (3.6), we have

$$(\phi_0, V_1(t_1, x(t_1, t_0, x_0), y(t_1, t_0, x_0))) \leqslant \frac{\delta}{2}. \quad (3.8)$$

From (3.4) and (3.7), we get

$$(\phi_0, V_{2\cdot\eta}(t_1, x(t_1, t_0, x_0), y(t_1, t_0, x_0))) \leqslant \frac{\delta}{2}. \quad (3.9)$$

Thus (3.3), (3.7)–(3.9), (H_6) yield the following contradiction:

$$\begin{aligned} b(\epsilon) &= b(\phi_0, x^*(t_1, t_0, x_0) + y^*(t_1, t_0, x_0)) \\ &\leqslant (\phi_0, V_{2\cdot\eta}(t_1, x^*(t_1, t_0, x_0)), y(t_1, t_0, x_0)) \\ &\leqslant a(\phi_0, x^*(t_2, t_0, x_0) + y^*(t_1, t_0, x_0)) \\ &= a(\delta) \leqslant b(\epsilon). \end{aligned}$$

Thus, the zero solution of (1.1) is ϕ_0-equistable, and the proof is completed. □

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